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# SOLUTION OF FUNCTIONAL EQUATIONS OF RESTRICTED $A^{(1)}_{n-1}$ FUSED LATTICE MODELS

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**Abstract**

Functional equations, in the form of fusion hierarchies, are studied for the transfer matrices of the fused restricted $A^{(1)}_{n-1}$ lattice models of Jimbo, Miwa and Okado. Specifically, these equations are solved analytically for the finite-size scaling spectra, central charges and some conformal weights. The results are obtained in terms of Rogers dilogarithm and correspond to coset conformal field theories based on the affine Lie algebra $A^{(1)}_{n-1}$ with GKO pair $A^{(1)}_{n-1} \oplus A^{(1)}_{n-1} \supset A^{(1)}_{n-1}$.

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# 1 Introduction

## 1.1 Conformal spectra and lattice models

In the last decade it has become clear that the critical behavior of two-dimensional statistical systems can be described by conformal field theories [1, 2, 3, 4]. This is possible because statistical systems at critical points possess conformal invariance [5]. In particular, the continuum limit of the critical $L$-state restricted solid-on-solid (RSOS) models of Andrews, Baxter and Forrester (ABF) [3] provide realizations of the unitary minimal conformal field theories with central charges

$$c \;=\; 1-\frac{6}{L(L+1)} \tag{1.1}$$

and conformal weights given by the Kac formula [6]

$$\Delta_{t,s} \;=\; \frac{[(L+1)t-Ls]^2-1}{4L(L+1)}\,, \quad 1\le t\le L-1\,, \quad 1\le s\le L\,, \quad s\le t\,. \tag{1.2}$$

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At criticality, the characters of the Virasoro algebra of the conformal field theory appear naturally in the modular invariant partition functions. In particular, the conformal spectra, including the central charge and the conformal weights, can be obtained from the finite-size corrections to the transfer matrix eigenvalues [7, 8]. Surprisingly, the same characters appear [3, 9] in the expressions for the off-critical local state probabilities of the ABF models.

Further conformal field theories are obtained from the fused ABF RSOS models at fusion level $p$. The central charge and conformal weights of these theories are given by

$$c = \frac{3p}{p+2} - \frac{6p}{(L+1)(L+1-p)}\,, \tag{1.3}$$

$$\Delta_{t,s} = \frac{[(L+1)t-(L+1-p)s]^2-p^2}{4p(L+1)(L+1-p)} + \frac{s_0(p-s_0)}{2p(p+2)} \tag{1.4}$$

where

$$0 \le s_0 \le p \text{ and } s_0 = \pm(t-s) \mod 2p\,. \tag{1.5}$$

This conformal spectra was conjectured [10, 11] by the evaluation of the local state probabilities and confirmed by the direct calculation [7, 8] of finite-size corrections for the fused ABF RSOS models. The fused lattice models are related to coset conformal field theories obtained by the Goddard-Kent-Olive (GKO) construction [12]. The relevant GKO pair is

$$\begin{array}{lccccc} & A_1^{(1)} & \oplus & A_1^{(1)} & \supset & A_1^{(1)} \\ \text{level} & L-p-1 & & p & & L-1 \end{array} \tag{1.6}$$

The minimal unitary series corresponds to level $p=1$. For $p>1$ the conformal characters are identified as branching coefficients.

Another direction to extend the unitary minimal series is to higher rank theories with the GKO pair

$$\begin{array}{lccccc} & A_{n-1}^{(1)} & \oplus & A_{n-1}^{(1)} & \supset & A_{n-1}^{(1)} \\ \text{level} & l-1 & & 1 & & l \end{array} \tag{1.7}$$

These theories reduce to the unitary minimal series for $n=2$ and $l=L-1$. The extended Virasoro algebra $\mathcal{W}_1^n$ can be constructed [13, 14] starting from this GKO pair of the affine algebra $A_{n-1}^{(1)}$. It has been confirmed that the integrable lattice models corresponding to these higher rank coset conformal field theories (1.8) are the $A_{n-1}^{(1)}$ lattice models of Jimbo, Miwa and Okado [15]. The central charge and conformal weights are

$$c = (n-1)\left[1-\frac{n(n+1)}{(l+n)(l+n-1)}\right] \tag{1.8}$$

$$\Delta_{\boldsymbol{t},\boldsymbol{s}} = \frac{[(l+n)\boldsymbol{t}-(l+n-1)\boldsymbol{s}]^2-n(n^2-1)/12}{2(l+n-1)(l+n)}\,, \tag{1.9}$$

where $\boldsymbol{t} = \mathbf{p} + \boldsymbol{\rho}$ and $\boldsymbol{s} = \mathbf{q} + \boldsymbol{\rho}$. Here $\boldsymbol{\rho}$ is the Weyl vector or the sum of all fundamental weights of $A^{(1)}_{n-1}$ and $\mathbf{p}$ and $\mathbf{q}$ are local states given respectively by the dominant integral weights $P_+(n, l-1)$ of level $l-1$ and $P_+(n, l)$ of level $l$. These notations are briefly explained in subsection 2.1. The conformal spectra (1.8) and (1.9) has been obtained by the GKO construction of the stress-energy in conformal field theory [16] and the study of local state probabilities of the JMO lattice models [15].

Further generalizing (1.7) and (1.8) by the fusion procedure leads to the GKO pair

$$\begin{array}{lccccc} & A^{(1)}_{n-1} & \oplus & A^{(1)}_{n-1} & \supset & A^{(1)}_{n-1} \\ \text{level} & l-p-1 & & p & & l \end{array} \tag{1.10}$$

with Virasoro algebra $\mathcal{W}^n_p$. Using the GKO construction of the stress-energy tensor, the central charge [16, 17, 18, 19] and conformal weights [16] of the $\mathcal{W}^n_p$ models are given by

$$c \;=\; \frac{(n^2-1)p}{p+n} - \frac{n(n^2-1)p}{(l+n)(l+n-p)} \tag{1.11}$$

$$\Delta_{\boldsymbol{t},\boldsymbol{s}} \;=\; \frac{[(l+n)\boldsymbol{t} - (l+n-p)\boldsymbol{s}]^2 - p^2 n(n^2-1)/12}{2p(l+n-p)(l+n)} + \delta_{\boldsymbol{\nu}} \;, \tag{1.12}$$

where $\boldsymbol{t} = \mathbf{p} + \boldsymbol{\rho}$, $\boldsymbol{s} = \mathbf{q} + \boldsymbol{\rho}$ and $\mathbf{p}$, $\mathbf{q}$ and $\boldsymbol{\nu}$ are local states respectively in the dominant integral weights $P_+(n, l-p)$, $P_+(n, l)$ and $P_+(n, p)$. By the Feigin-Fuchs construction [21, 22], $\delta_{\boldsymbol{\nu}}$ is fixed by [16]

$$\delta_{\boldsymbol{\nu}} \;=\; \frac{2p\boldsymbol{\nu}\cdot\boldsymbol{\rho} - n\boldsymbol{\nu}^2}{2p(p+n)} \;. \tag{1.13}$$

In this paper we calculate the finite-size corrections to the eigenvalue spectra of the transfer matrices of the fused JMO $A^{(1)}_{n-1}$ lattice models. Specifically, we generalize the analytic study introduced in [8, 19] to obtain the central charges and conformal weights of the fused JMO $A^{(1)}_{n-1}$ lattice models. The functional equations of the fused transfer matrices for the JMO $A^{(1)}_{n-1}$ lattice model have been given in [17]. In [20] the thermodynamic Bethe ansatz-like equations (also called $y$-systems or inversion identity hierarchies) of the model have been introduced and by solving these functional equations Kuniba, Nakanishi and Suzuki have extracted the central charge (1.11) from the finite-size corrections to the eigenvalue spectra of the transfer matrices. In fact the central charge and conformal weights together appear in the finite-size corrections to the eigenvalue spectra of the transfer matrices. To obtain the conformal weights we have to consider the excited states of the transfer matrices and the calculations are more complicated because of the more complicated analyticity for the states. Therefore we generalize the study done in [18, 19] for the fused JMO $A^{(1)}_{n-1}$ lattice models to calculate some conformal weights. In the scaling limit the fused JMO models have been shown [17] to yield the same central charges (1.11) as the $\mathcal{W}^n_p$ conformal field theories. Here we find the conformal weights

$$\Delta_{t,s} \;=\; \frac{n(n^2-1)}{24}\,\frac{[(l+n)t - (l+n-p)s]^2 - p^2}{p(l+n-p)(l+n)} + \frac{2p\nu - n\nu^2}{2p(p+n)} \;, \tag{1.14}$$

where $1 \le s < \left\lfloor \frac{l}{n-1} \right\rfloor$, $1 \le t < \left\lfloor \frac{l-p}{n-1} \right\rfloor$ and $\nu = (s-t) - \left\lfloor \frac{s-t}{p} \right\rfloor p$ , where $\lfloor x \rfloor$ denotes the largest less than or equal to x. Clearly, (1.12) and (1.14) agree if

$$\begin{aligned} \langle \boldsymbol{t}, \boldsymbol{s} \rangle &= \frac{n(n^2-1)}{12} ts \, , \\ \boldsymbol{t}^2 &= \frac{n(n^2-1)}{12} t^2 \, , \\ \boldsymbol{s}^2 &= \frac{n(n^2-1)}{12} s^2 \, , \\ \boldsymbol{\nu}^2 &= \frac{n(n^2-1)}{12} \nu^2 \, , \\ \langle \boldsymbol{\nu}, \boldsymbol{\rho} \rangle &= \frac{n(n^2-1)}{12} \nu \end{aligned} \tag{1.15}$$

This confirms that the underlying solvable statistical mechanics models corresponding to the conformal field theories with extended algebra $\mathcal{W}_p^n$ are precisely the fused critical $A_{n-1}^{(1)}$ lattice models of Jimbo, Miwa and Okado.

The paper is organized as follows. In the next subsection, we describe the finite-size corrections to the eigenvalues of the row transfer matrices of critical lattice models as predicted by conformal invariance. In section 2 we define the $A_{n-1}^{(1)}$ lattice models of Jimbo, Miwa and Okado. In particular, we discuss the fusion rules satisfied by the adjacency matrices of these models and the corresponding functional equations of fused transfer matrices. These functional equations can be converted into inversion identity hierarchies. These inversion identity hierarchies and their original functional equations are the elementary relations needed to determine the finite-size corrections to the eigenvalues of the row transfer matrices. Next we find the asymptotic and bulk behavior of these inversion identity hierarchies, which are the important data for obtaining finite-size corrections. In section 3, we convert these functional equations into nonlinear integral equations which we solve analytically. After using some known [23] Rogers dilogarithm identities, we obtain the finite-size corrections, the central charges and conformal weights. Finally, a brief conclusion is given in section 4.

## 1.2 Finite-size corrections

Much work has been done on extracting the conformal spectra of exactly solvable lattice models from finite-size corrections. These extensive calculations [24, 25, 26, 27, 28, 29, 30, 7, 31, 17, 32, 8] give very strong and direct evidence to support the predictions of conformal and modular invariance.

According to the predictions of conformal and modular invariance, the partition function of a two-dimensional lattice model on a finite $M \times N$ periodic lattice or torus, can be written for large $M$ and $N$ as

$$Z_{M,N} = Tr\ \boldsymbol{T}^M \sim Z(q) e^{-MNf} \, . \tag{1.16}$$

Here $\boldsymbol{T}$ is the row transfer matrix, $f$ is the bulk free energy and $Z(q)$ is the universal

finite-size partition function with modular parameter

$$q = e^{2\pi i\tau} \ , \qquad \tau = \frac{M}{N}\, e^{i(\pi - hu)} \ . \tag{1.17}$$

Suppose that the eigenvalues of the row transfer matrix $\boldsymbol{T}$ of a periodic row of $N$ faces are given by

$$\Lambda_n = e^{-E_n} \tag{1.18}$$

where $E_n$ are the corresponding energy levels. It then follows that

$$Z_{M,N} = \sum_n e^{-ME_n}. \tag{1.19}$$

Conformal invariance now predicts [33, 34] that for large N the energy levels take the form

$$E_0 \ \sim \ Nf - \frac{\pi c}{6N}\sin(hu) \ , \tag{1.20}$$

$$E_n \ \sim \ E_0 + \frac{2\pi}{N}[x_n \sin(hu) + is_n \cos(hu)] \tag{1.21}$$

where $E_0$ is the ground-state energy,

$$x_n = \Delta + \overline{\Delta} + k + \overline{k} \ \text{ and } \ s_n = \Delta - \overline{\Delta} + k - \overline{k} \tag{1.22}$$

are respectively the scaling dimensions and spins of the various levels and $k, \overline{k}$ are integers. The numbers $c$ and $(\Delta, \overline{\Delta})$ are identified as the central charge and conformal weights of the primary operators of the underlying conformal field theory.

# 2 The models and their fusion hierarchies

## 2.1 Algebraic notations and Boltzmann weights

A vector $a$ represents a level $l$ dominant integral weight of $A^{(1)}_{n-1}$ if

$$a = \sum_{\mu=0}^{n-1} a_\mu \Lambda_\mu, \qquad a_\mu \in \mathbb{Z}_+ \tag{2.1}$$

and $\sum_{\mu=0}^{n-1} a_\mu = l$, where $\mathbb{Z}_+$ is a set of all non-negative integers and $\Lambda_\mu$ with $\mu = 0, 1, \cdots, n-1$ are the fundamental weights of $A^{(1)}_{n-1}$ with $\Lambda_n = \Lambda_0$. Fix an integer $l \geq 1$, and denote by $P_+(n,l)$ the set of dominant weights. Then an element of $P_+(n,l)$ is called a local state.

An ordered pair of local states $(a, b)$, with $a, b \in P_+(n,l)$, is called admissible if

$$b = a + \hat{\mu}, \qquad \mu = 0, 1, \cdots, n-1 \tag{2.2}$$

where $\hat{\mu}$ are the elementary vectors defined by

$$\hat{\mu} = \Lambda_{\mu+1} - \Lambda_\mu \qquad \mu = 0, 1, \cdots, n-1 \ . \tag{2.3}$$

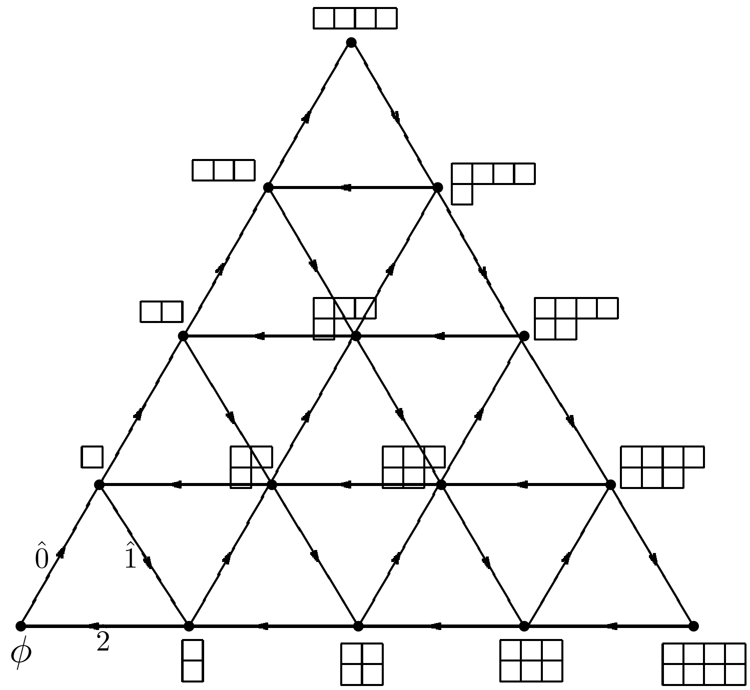



Figure 1: The set of local states $P_+(3,4)$ ($n=3$, $l=4$). Each state is associated with a corresponding Young diagram and the admissible pairs of the states $(a,b)$ are represented by arrows from $a$ to $b$.

For each such pair we place an arrow from $a$ to $b$, then all local states in set $P_+(n,l)$ can be represented by an oriented graph. There is a one-to-one correspondence between a state in $P_+(n,l)$ and a Young diagram $(f_1,\cdots,f_{n-1})$ with $l=f_0 \geq f_1 \geq \cdots \geq f_{n-1} \geq f_n = 0$ (see Figures 1 and 2) given by

$$a = \sum_{\mu=0}^{n-1} (f_\mu - f_{\mu+1})\Lambda_\mu.$$

It is usual to omit the columns of length $n$ in these Young diagrams.

The $A^{(1)}_{n-1}$ lattice models are described by assigning a Boltzmann weight

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) = \begin{array}{ccc} d & & c \\ & \boxed{u} & \\ a & & b \end{array}$$

to each configuration $a,b,c,d \in P_+(n,l)$ of four sites surrounding a face. The face weights are nonzero only if $(d,a),(a,b),(d,c),(c,b)$ are all admissible. The nonzero weights are

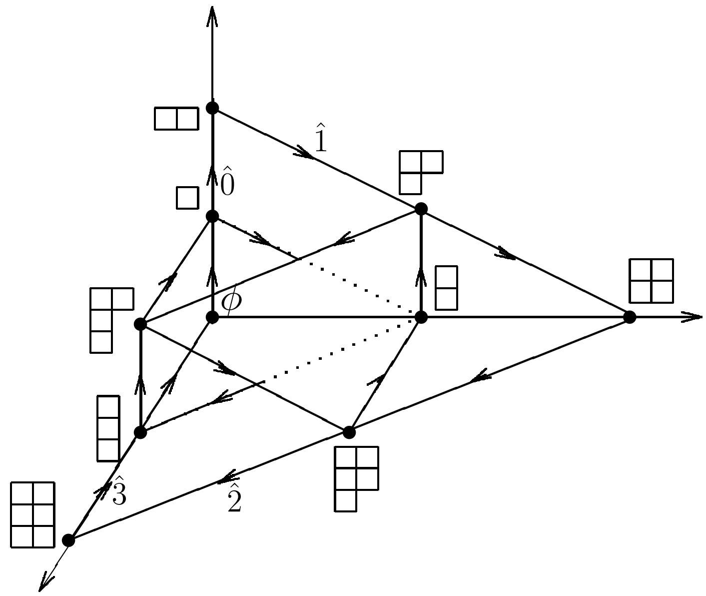



Figure 2: The set of local states $P_+(4,2)$ ($n=4$, $l=2$). Each state is associated with a corresponding Young diagram and the admissible pairs of the states $(a,b)$ are represented by the arrows from $a$ to $b$.

given by [35]

$$W\left(\begin{matrix} a & a+\hat{\alpha} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} \end{matrix}\middle| u\right) = \begin{cases} \dfrac{h(\lambda+u)}{h(\lambda)}, & \text{if } 0 \le \alpha=\mu=\nu \le n-1; \\[2ex] \dfrac{h(\lambda a^{\mu\nu}-u)}{h(\lambda a^{\mu\nu})}, & \text{if } 0 \le \alpha=\mu\neq\nu \le n-1; \\[2ex] \dfrac{h(u)}{h(\lambda)}\dfrac{h(\lambda a^{\mu\nu}+\lambda)}{h(\lambda a^{\mu\nu})}, & \text{if } 0 \le \alpha=\nu\neq\mu \le n-1, \end{cases} \tag{2.4}$$

where $a$ is a local state and $a^{\mu\nu}$ is given by the inner-product $\langle a+\rho, \hat{\mu}-\hat{\nu}\rangle$. Here $u$ is the spectral parameter, $\lambda = \frac{\pi}{n+l}$, $\rho$ is the sum over all fundamental weights $\Lambda_\mu$ and $h(u)$ is the elliptic theta function

$$h(u) = \sin(u)\prod_{j=1}^{\infty}[(1-2q^{2j}\cos 2u + q^{4j})(1-q^{2j})]. \tag{2.5}$$

In this paper we will only consider the critical case when the elliptic nome vanishes $q=0$ and the theta function reduces to the trigonometric function

$$h(u) = \sin(u). \tag{2.6}$$

## 2.2 su($n$) fusion rules

The oriented graphs, such as that shown in Figure 1 and 2, describe all admissible pairs $(a, b)$ in $P_+(n, l)$. The corresponding adjacency matrix $A$ with elements

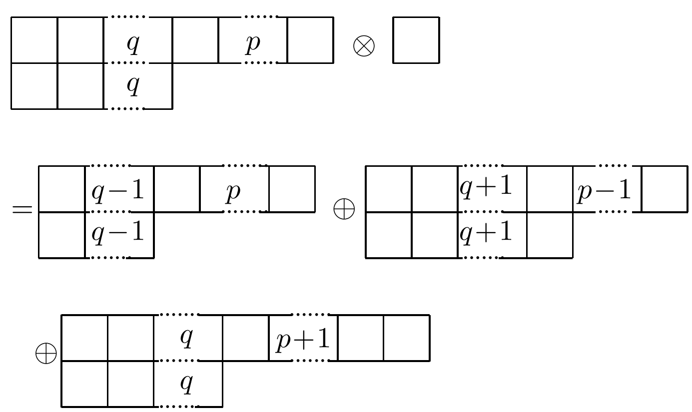
$$A_{a,b} = \begin{cases} 1 & (a, b) \text{ admissible} \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

satisfies certain $su(n)$ fusion rules which determine the admissible pairs $(a, b)$ in $P_+(n, l)$ for the fused $A^{(1)}_{n-1}$ lattice models as we will now explain.

For fixed rank $n \geq 2$ and level $l$, the $su(n)$ fusion rules determine a hierarchy of commuting adjacency matrices labeled by the representations of $su(n)$. Set $\omega_1 = (1, 0, \cdots, 0)$ and consider the tensor product of two irreducible representations of of $su(n)$ with Young tableaux $f = (f_1, f_2, \cdots, f_{n-1})$ and $\omega_1$. Suppose the decomposition of the tensor product gives $s$ irreducible representations with Young tableaux $f^k = (f^k_1, f^k_2, \cdots, f^k_{n-1})$ $k = 1, 2, \cdots s$ with $s < n$. Then the $su(n)$ fusion rule is expressed as

$$A^{(f)} A = \sum_{k=1}^{s} A^{(f^k)} , \tag{2.8}$$

where $A^{(f)} = \mathrm{I}$ if $f = (0, 0, \cdots, 0)$ and $A^{(f)} = 0$ unless $l \geq f_1 \geq \cdots \geq f_{n-1} \geq f_n = 0$. In the oriented adjacency graphs $s$ is just the number of outgoing arrows originating from the local state with Young tableau $f$ and $f^1, f^2, \cdots f^s$ are the corresponding adjacent local states to which the outgoing arrows point (e.g. see Figure 1). For example, suppose that $n = 3$ and $A^{(f)}$ is represented by the Young diagram $f = (f_1, f_2, \cdots, f_{n-1})$. Then the fusion rule (2.8) takes the form




where $p = f_1 - f_2$ and $q = f_2$. There is a level-rank duality of the adjacency matrices. Let $\overline{f}$ be the Young tableaux obtained from $f$ by transposing the diagram. Then the adjacency matrix $A^{(f)}$ of models with rank $n$ and level $l$ is the same as the adjacency matrix $A^{(\overline{f})}$ of models with rank $l + 1$ and level $n - 1$.

Of particular interest here are weights $p\Lambda_a$ in $P_+(n, l)$ with rectangular Young tableaux $f = (f_1, f_2, \cdots, f_{n-1})$ where $f_{s \leq a} = p$ and $f_{s>a} = 0$. Let us write the corresponding adjacency matrix as $A^{(a,p)}$. Then, from the fusion rule (2.8), it follows that

$$A^{(a,p)} A^{(a,p)} = A^{(a,p-1)} A^{(a,p+1)} + A^{(a-1,p)} A^{(a+1,p)} , \tag{2.9}$$

where $a = 1, 2, \cdots, n-1$ and $p = 1, 2, \cdots, l$. Here $A^{(a,p)} = \mathrm{I}$ if $a = 0$ or $p = 0$ and the closure condition is that $A^{(a,p)} = 0$ if $p < 0$ or $p > l$ or $a < 0$ or $a > n$. In the next subsection 2.3 we will see that a similar relation holds at the level of the transfer matrices of the fused models.

In general, the admissible states of adjacent sites of the fused models are given by the $su(n)$ fusion rule. In the appendix we give the explicit solution of the fusion rules for $n = 4$ and $l = 2$. In contrast to fusing the ABF models, the elements of $A^{(f^k)}$ for $n > 2$ can in general be nonnegative integers greater than one, for example, $A^{(3,1)}$ for $n = 3$ and $l = 5$. In such cases we have to distinguish the oriented edges between the adjacency states $(a, b)$ by bond variables $\alpha = 1, 2, \ldots, A^{(f)}_{a,b}$.

## 2.3 Functional equations

A class of fused $A^{(1)}_{n-1}$ lattice models with adjacency matrices $A^{(f)}$ can be constructed by the fusion procedure [36]. Let us consider the fused models obtained by fusing $(a\ p) \times (b\ q)$ elementary blocks corresponding to the representations $p\Lambda_a$ and $q\Lambda_b$. The fused face weights of these models vanish, that is,

$$W^{(b,q)}_{(a,p)}\left(\begin{array}{ccc|c} d & \nu & c & \\ \alpha & & \beta & u \\ a & \mu & b & \end{array}\right) = \begin{array}{ccc} d & \nu & c \\ \alpha & \boxed{\;u\;} & \beta \\ a & \mu & b \end{array} = 0$$

unless $A^{(a,p)}_{d,a} \neq 0$, $A^{(a,p)}_{c,b} \neq 0$, $A^{(b,q)}_{a,b} \neq 0$ and $A^{(b,q)}_{d,c} \neq 0$ and $1 \leq \alpha \leq A^{(a,p)}_{d,a}$, $1 \leq \beta \leq A^{(a,p)}_{c,b}$, $1 \leq \mu \leq A^{(b,q)}_{a,b}$ and $1 \leq \nu \leq A^{(b,q)}_{d,c}$. Suppose that $\boldsymbol{a}(\boldsymbol{\alpha})$ and $\boldsymbol{b}(\boldsymbol{\beta})$ are allowed state (bond) configurations of two consecutive rows of a lattice with $N$ columns and periodic boundary conditions. The elements of the fused row transfer matrices $\mathbf{T}^{(b,q)}_{(a,p)}(u)$ are given by

$$\langle \boldsymbol{a}, \boldsymbol{\alpha} | \mathbf{T}^{(b,q)}_{(a,p)}(u) | \boldsymbol{b}, \boldsymbol{\beta} \rangle = \prod_{j=1}^{N} \sum_{\{\eta_j\}} W^{(b,q)}_{(a,p)}\left(\begin{array}{ccc|c} a_{j+1} & \eta_{j+1} & b_{j+1} & \\ \alpha_j & & \beta_j & u \\ a_j & \eta_j & b_j & \end{array}\right) = \begin{array}{ccc} a_{j+1} & & b_{j+1} \\ \alpha_j & \boxed{\;u\;} & \beta_j \\ a_j & & b_j \end{array} \tag{2.10}$$

where $a_{N+1} = a_1$, $b_{N+1} = b_1$, $\eta_{N+1} = \eta_1$ and $N = 0 \bmod n$. Since the fused face weights satisfy the Yang-Baxter relation, we obtain a hierarchy of commuting families of transfer matrices. Specifically, if $(a, p)$ are held fixed

$$[\mathbf{T}^{(b,q)}(u), \mathbf{T}^{(b',q')}(v)] = 0, \tag{2.11}$$

where we have suppressed the subscripts $\boldsymbol{T}^{(b,q)}(u) = \boldsymbol{T}^{(b,q)}_{(a,p)}(u)$. These transfer matrices satisfy a group of functional equations which are expressed in determinant form in [17]. From this determinant form the following useful functional equations can be extracted [20],

$$\begin{aligned} \boldsymbol{T}^{(b,q)}(u)\ \boldsymbol{T}^{(b,q)}(u-\lambda) \ &= \\ \boldsymbol{T}^{(b,q+1)}(u)\ \boldsymbol{T}^{(b,q-1)}(u-\lambda) + \ &\boldsymbol{T}^{(b+1,q)}(u)\ \boldsymbol{T}^{(b-1,q)}(u-\lambda) \end{aligned} \tag{2.12}$$

These functional equations can be established using the fusion procedure [36] and they are the Baxterization of the $su(n)$ fusion rule (2.9) for the adjacency matrices.

The functional equations are easily converted into the so called thermodynamic Bethe ansatz-like (TBA-like) equations in the sense of [20, 37]. Thus inserting

$$\boldsymbol{t}^{(b,q)}(u) := \frac{\boldsymbol{T}^{(b,q+1)}(u)\ \boldsymbol{T}^{(b,q-1)}(u-\lambda)}{\boldsymbol{T}^{(b+1,q)}(u)\ \boldsymbol{T}^{(b-1,q)}(u-\lambda)} \tag{2.13}$$

into (2.12) yields the TBA-like hierarchy

$$\boldsymbol{t}^{(b,q)}(u)\boldsymbol{t}^{(b,q)}(u-\lambda) = \frac{[\mathrm{I}+\boldsymbol{t}^{(b,q+1)}(u)][\mathrm{I}+\boldsymbol{t}^{(b,q-1)}(u-\lambda)]}{[\mathrm{I}+(\boldsymbol{t}^{(b+1,q)}(u))^{-1}][\mathrm{I}+(\boldsymbol{t}^{(b-1,q)}(u-\lambda))^{-1}]} \tag{2.14}$$

The closure condition becomes

$$\boldsymbol{t}^{(b,0)}(u) = \boldsymbol{t}^{(b,l)}(u) = 0\ , \quad b = 1, 2, \cdots, n-1\ . \tag{2.15}$$

By definition we have the following symmetry. Applying the replacement

$$\boldsymbol{t}^{(b,q)}(u) \to \tilde{\boldsymbol{t}}^{(b,q)}(u) = \frac{1}{\boldsymbol{t}^{(q,b)}(u)} \tag{2.16}$$

to the TBA-like hierarchy, we have

$$\tilde{\boldsymbol{t}}^{(b,q)}(u)\tilde{\boldsymbol{t}}^{(b,q)}(u-\lambda) = \frac{[\mathrm{I}+\tilde{\boldsymbol{t}}^{(b,q+1)}(u)][\mathrm{I}+\tilde{\boldsymbol{t}}^{(b,q-1)}(u-\lambda)]}{[\mathrm{I}+(\tilde{\boldsymbol{t}}^{(b+1,q)}(u))^{-1}][\mathrm{I}+(\tilde{\boldsymbol{t}}^{(b-1,q)}(u-\lambda))^{-1}]} \tag{2.17}$$

with the closure condition

$$\tilde{\boldsymbol{t}}^{(b,0)}(u) = \tilde{\boldsymbol{t}}^{(b,n)}(u) = 0\ , \quad b = 1, 2, \cdots, l-1\ . \tag{2.18}$$

This symmetry is just level-rank duality [38, 39].

In terms of the $\boldsymbol{t}$ matrices, the functional equations (2.12) can be rewritten as

$$\boldsymbol{T}^{(a,p)}(u)\ \boldsymbol{T}^{(a,p)}(u-\lambda) = \ \boldsymbol{T}^{(a-1,p)}(u-\lambda)\ \boldsymbol{T}^{(a+1,p)}(u)\Big(1+\boldsymbol{t}^{(a,p)}(u)\Big) \tag{2.19}$$

It is obvious to see that we need to solve the TBA-like equations (2.14) first to get the solutions for the transfer matrices $\boldsymbol{T}^{(a,p)}(u)$.

## 2.4 Asymptotics of $t$

The functional equations (2.14) are easily solved in the braid limit $u \to \pm i\infty$. The asymptotics $t^{(b,q)}_\infty = t^{(b,q)}(\pm i\infty)$ satisfy

$$t^{(b,q)}_\infty t^{(b,q)}_\infty = \frac{[1+t^{(b,q+1)}_\infty][1+t^{(b,q-1)}_\infty]}{[1+(t^{(b+1,q)}_\infty)^{-1}][1+(t^{(b-1,q)}_\infty)^{-1}]} \tag{2.20}$$

with the closure condition

$$t_{\infty}^{(b,0)} = t_{\infty}^{(b,l)} = 0 \ , \qquad b = 1, 2, \cdots, n-1 \ . \tag{2.21}$$

Let us write $t_{\infty}^{(1,1)}$ as

$$t_{\infty}^{(1,1)} = \frac{\sin[(n+1)\theta]}{\sin[(n-1)\theta]} \tag{2.22}$$

with $\theta$ to be determined. Using (2.20) as a recursion relation for $t_{\infty}^{(b,q)}$ we then derive

$$t_{\infty}^{(b,q)} = \frac{\sin[(n+q)\theta]\sin[q\theta]}{\sin[(n-b)\theta]\sin[b\theta]} \tag{2.23}$$

for $b = 1, 2, \cdots, n-1$ and $q = 1, 2, \cdots, l-1$. The closure condition (2.21) imposes the quantization

$$\theta = \frac{m_j \pi}{h} \ , \qquad m_j = 1, 2, \cdots, \left\lfloor \frac{l}{n-1} \right\rfloor , \tag{2.24}$$

where $h = n + l$. Mathematically, the exponent $m_j$ can go from 1 to $h - 1$. But $t_{\infty}^{(b,q)}$ with $m_j > \left\lfloor \frac{l}{n-1} \right\rfloor$ is on longer the right solutions of the models. This can be checked by looking the adjacency matrices or the transfer matrices for small size.

We would like to mention here that the solutions given by (2.23) are not the general ones of the equations (2.20). However, the general solutions involve complex eigenvalues, which together with the real eigenvalues (2.23) give all of the low-lying excited states.

## 2.5 Zeros and poles of the transfer matrices

To find the finite-size correction we need to solve the TBA equations (2.14) or (2.17) in certain analytic domains. Inside these analyticity strips the ground state eigenvalues $T^{(b,q)}(u) = T^{(b,q)}_{(a,p)}(u)$ should not possess any zero except those which come from the parameterization of Boltzmann weights. They are of order $N$ and their locations are independent of the eigenvalue under consideration. Using self explanatory notation, the location of these zeros are as follows:

$$\bigcup_{j=0}^{a-1}\bigcup_{l=0}^{p-1}\bigcup_{i=0}^{b-1}\bigcup_{k=0}^{q-2}\{(i+k-j-l)\lambda\}\bigcup_{j=0}^{a-1}\bigcup_{l=0}^{p-1}\bigcup_{i=0}^{b-2}\{(q+i-j-l)\lambda\} \ , \quad q \geq p \quad \text{and} \quad b \geq a \tag{2.25}$$

$$\bigcup_{j=0}^{a-1}\bigcup_{l=0}^{p-1}\bigcup_{i=0}^{b-1}\bigcup_{k=0}^{q-2}\{(i+k-j-l)\lambda\}\bigcup_{j=1}^{a-1}\bigcup_{l=0}^{p-1}\bigcup_{i=0}^{b-1}\{(q+i-j-l)\lambda\} \ , \quad q \geq p \quad \text{and} \quad b \leq a \tag{2.26}$$

$$\bigcup_{j=0}^{a-1}\bigcup_{l=1}^{p-1}\bigcup_{i=0}^{b-1}\bigcup_{k=0}^{q-1}\{(i+k-j-l)\lambda\}\bigcup_{i=1}^{b-1}\bigcup_{k=0}^{q-1}\bigcup_{j=0}^{a-1}\{(k+i-j)\lambda\} \ , \quad q \leq p \quad \text{and} \quad b \geq a \tag{2.27}$$

$$\bigcup_{j=0}^{a-1}\bigcup_{l=1}^{p-1}\bigcup_{i=0}^{b-1}\bigcup_{k=0}^{q-1}\{(i+k-j-l)\lambda\}\bigcup_{i=0}^{b-1}\bigcup_{k=0}^{q-1}\bigcup_{j=0}^{a-2}\{(k+i-j)\lambda\} \ , \quad q \leq p \quad \text{and} \quad b \leq a \tag{2.28}$$

These can be seen from the fusion procedure of the models [36]. To locate the zeros and poles of $t^{(b,q)}$ we use the definition (2.13) and the zeros of $T^{(b,q)}(u)$. We distinguish several cases as follows:

$$\begin{array}{rl}
\text{(I)} & 1 \le q \le p-1 \quad \text{and} \quad b=a \\
& \text{zeros: } \emptyset \\
& \text{poles: } \displaystyle\bigcup_{l=1}^{q}\{l\lambda\}
\end{array} \tag{2.29}$$

$$\begin{array}{rl}
\text{(II)} & q=p \quad \text{and} \quad 1 \le b \le a-1 \\
& \text{zeros: } \displaystyle\bigcup_{j=1}^{b}\{(j-a)\lambda\} \\
& \text{poles: } \emptyset
\end{array} \tag{2.30}$$

$$\begin{array}{rl}
\text{(III)} & q=p \quad \text{and} \quad b=a \\
& \text{zeros: } \displaystyle\bigcup_{j=0}^{a-1}\{-j\lambda\} \\
& \text{poles: } \displaystyle\bigcup_{l=1}^{p}\{\, l\lambda \,\}
\end{array} \tag{2.31}$$

$$\begin{array}{rl}
\text{(IV)} & q \ge p+1 \quad \text{and} \quad b=a \\
& \text{zeros: } \emptyset \\
& \text{poles: } \displaystyle\bigcup_{l=1}^{p}\{(l+q-p)\lambda\}
\end{array} \tag{2.32}$$

$$\begin{array}{rl}
\text{(V)} & q=p \quad \text{and} \quad b \ge a+1 \\
& \text{zeros: } \displaystyle\bigcup_{j=1}^{a}\{(j-a)\lambda\} \\
& \text{poles: } \emptyset
\end{array} \tag{2.33}$$

$$\begin{array}{rl}
\text{(VI)} & q \ne p \quad \text{and} \quad b \ne a \\
& \text{zeros: } \emptyset \\
& \text{poles: } \emptyset
\end{array} \tag{2.34}$$

This pattern of zeros and poles is divided according to the fusion level $(a,p)$ in the vertical direction. We fix the vertical fusion level $(a,p)$ and vary the fusion level $(b,q)$ in the horizontal direction. The transfer matrices $\boldsymbol{t}^{(b,q)}$ are free of zeros for $q \ne p$ and free of poles for $b \ne a$.

## 2.6 Bulk behavior and the largest eigenvalues

According to section 2.5 the analyticity strip for $t^{(a,p)}(u)$ contains a zero of order $N$ at $u=0$ and a poles of order $N$ at $u=\lambda$. All other functions $t^{(b,q)}$ are analytic and non-zero in their analyticity strips $-\frac{1}{2}\lambda \le u \le \frac{1}{2}\lambda$. For large $N$ the leading bulk behavior to $t^{(b,q)}$

is given by

$$t_{\rm bulk}^{(b,q)}(u) = \begin{cases} \text{constant} \;, & q \neq p \quad \text{or} \quad b \neq q \;, \\ \text{constant}\Big[\tan(\tfrac{1}{2}hu)\Big]^N \;, & q = p \;\; \text{and} \;\; b = a \;. \end{cases} \tag{2.35}$$

The ansatz of bulk behavior matches the zero and pole distribution. The constants are fixed by the TBA-like equations (2.14) and can be calculated similarly to the asymptotics of $t^{(b,q)}$. Corresponding to the asymptotics solutions (2.23) we find that the bulk values $t_{\rm bulk}^{(b,q)}$ for $1 \le q \le p-1$ and $1 \le b \le n-1$ are given by

$$t_{\rm bulk}^{(b,q)} = \frac{\sin[(q+n)\sigma]\sin(q\sigma)}{\sin(b\sigma)\sin[(n-b)\sigma]} \tag{2.36}$$

with

$$\sigma = \frac{m_j'\pi}{p+n} \qquad m_j' = 1, 2, \cdots, \left\lfloor \frac{p}{n-1} \right\rfloor \tag{2.37}$$

and for $p+1 \le q \le h-n-1$ and $1 \le b \le n-1$ by

$$t_{\rm bulk}^{(b,q)} = \frac{\sin[(q-p+n)\tau]\sin[(q-p)\tau]}{\sin(b\tau)\sin[(n-b)\tau]} \tag{2.38}$$

with

$$\tau = \frac{m_j''\pi}{h-p} \qquad m_j'' = 1, 2, \cdots, \left\lfloor \frac{l-p}{n-1} \right\rfloor . \tag{2.39}$$

Here we see that $p = 1, 2, \cdots, h-2$. For the largest eigenvalue, the appropriate choices are $\theta = \pi/h$, $\sigma = \pi/(p+n)$ and $\tau = \pi/(h-p)$ in (2.23), (2.36) and (2.38) respectively. Here for the same reason as the braid limit (2.24) we restrict the range of the exponents $m_j^{'}, m_j^{''}$.

# 3 Finite-size corrections

The finite-size corrections for the eigenvalues $T^{(a,p)}$ can be obtained by solving the equation

$$T^{(a,p)}(u)T^{(a,p)}(u-\lambda) = T^{(a-1,p)}(u-\lambda)T^{(a+1,p)}(u)\Big(1+t^{(a,p)}(u)\Big) \tag{3.1}$$

Although the JMO $A_{n-1}^{(1)}$ models satisfy an inversion relation, there is no crossing symmetry for the face weights of these models so we cannot extract the bulk behavior for the transfer matrices from the inversion relation alone. Instead the finite-size correction to $T^{(a,p)}(u)$ is given by

$$T^{(a,p)}(u) = T_{\rm bulk}^{(a,p)}(u)T_{\rm finite}^{(a,p)}(u) \;. \tag{3.2}$$

Inserting (3.2) into (3.1) and setting the bulk behavior as

$$T^{(a,p)}_{\text{bulk}}(u)T^{(a,p)}_{\text{bulk}}(u-\lambda) = T^{(a-1,p)}(u-\lambda)T^{(a+1,p)}(u) \tag{3.3}$$

we find

$$T^{(a,p)}_{\text{finite}}(u)T^{(a,p)}_{\text{finite}}(u-\lambda) = 1 + t^{(a,p)}(u) \ . \tag{3.4}$$

We can check that (3.3) is correct for the case of $n=2$ or the $(l+1)$-state ABF models in [8]. The finite-size corrections for $T^{(a,p)}(u)$, therefore, are represented by the inversion identity hierarchy $t^{(a,p)}(u)$. In the analytical treatment of (3.4) and (2.14), we will see that the only inputs for the finite-size corrections in the scaling limit are the asymptotics and bulk behavior.

## 3.1 Nonlinear equations for finite-size corrections

It is useful to introduce functions of a real variable by restricting the eigenvalue functions to certain lines in the complex plane,

$$\begin{aligned}
\mathfrak{A}^{(b,q)}(x) &:= 1 + \mathfrak{a}^{(b,q)}(x) \ , &(3.5)\\
\mathfrak{a}^{(b,q)}(x) &:= t^{(b,q)}\Big(\frac{ni}{2h}x - \frac{a-b+p-q}{2}\lambda\Big) \ , &(3.6)\\
\mathfrak{b}^{(b,q)}(x) &:= T^{(b,q)}_{\text{finite}}\Big(\frac{ni}{2h}x - \frac{a-b+p-q+1}{2}\lambda\Big) \ . &(3.7)
\end{aligned}$$

The functional relation (3.4) can then be rewritten in terms of the new functions as

$$\mathfrak{b}^{(a,p)}(x-\pi i/n)\mathfrak{b}^{(a,p)}(x+\pi i/n) = \mathfrak{A}^{(a,p)}(x) \ . \tag{3.8}$$

For the ground state the functions $\mathfrak{A}^{(b,q)}$ and $\mathfrak{b}^{(b,q)}$ are *analytic, non-zero* in $-2\pi/n < x < 2\pi/n$ and possess *constant* asymptotics for $\mathcal{R}e\ x \to \pm\infty$ (ANZC). Taking the logarithmic derivative of the above equation and introducing Fourier transforms

$$\begin{aligned}
\mathcal{B}^{(a,p)}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ [\ln \mathfrak{b}^{(a,p)}(x)]'\ e^{-ikx} \ ,\\
[\ln \mathfrak{b}^{(a,p)}(x)]' &= \int_{-\infty}^{\infty} dk\ \mathcal{B}^{(a,p)}(k)\ e^{ikx}
\end{aligned} \tag{3.9}$$

with analogous equations for $\mathfrak{A}^{(a,p)}$ and its Fourier transform $A^{(a,p)}$, we then find that

$$\mathcal{B}^{(a,p)}(k) = \frac{A^{(a,p)}(k)}{e^{(\pi/n)k} + e^{-(\pi/n)k}} \ . \tag{3.10}$$

Transforming back and defining the kernel $k(x)$

$$k(x) := \frac{n}{4\pi\cosh(nx/2)} \ , \tag{3.11}$$

we are able to express $\mathfrak{b}^{(a,p)}$ in terms of $\mathfrak{A}^{(a,p)}$,

$$\ln \mathfrak{b}^{(a,p)} = k * \ln \mathfrak{A}^{(a,p)} + C^{(a,p)} \ , \tag{3.12}$$

where $C^{(a,p)}$ are integration constants. The convolution $f * g$ of two functions $f$ and $g$ is defined by

$$(f * g) := \int_{-\infty}^{\infty} f(x-y)g(y)\, dy = \int_{-\infty}^{\infty} g(x-y)f(y)\, dy \; . \tag{3.13}$$

In the case of low-lying excitations we have to take care of the zeros in the analyticity strips so that the simple ANZC properties hold. The result (3.12) is still correct if we change the integration path $\mathcal{L}$ so that $b^{(a,p)}(x)$ has an ANZC area and Cauchy's theorem can be applied as in [8]. The integration constants in (3.12) can be evaluated from the asymptotics of $\mathfrak{U}^{(a,p)}$ and $\mathfrak{b}^{(a,p)}$. In this limit (3.12) becomes

$$\ln \mathfrak{b}_{\infty}^{(a,p)} = \tfrac{1}{2} \ln \mathfrak{U}_{\infty}^{(a,p)} + C^{(a,p)} \; . \tag{3.14}$$

It can be seen that the constants are dependent on the system size $N$ and do not contribute to the $\frac{1}{N}$ corrections.

The $\mathfrak{U}^{(a,p)}$ is from the inversion identity hierarchy and can be solved from (2.14), which can be rewritten in terms of $\mathfrak{a}^{(b,q)}$ as

$$\frac{\mathfrak{a}^{(b,q)}(x-\pi i/n)\mathfrak{a}^{(b,q)}(x+\pi i/n)}{\mathfrak{a}^{(b-1,q)}(x)\mathfrak{a}^{(b+1,q)}(x)} = \frac{\mathfrak{U}^{(b,q-1)}(x)\mathfrak{U}^{(b,q+1)}(x)}{\mathfrak{U}^{(b-1,q)}(x)\mathfrak{U}^{(b+1,q)}(x)} \; . \tag{3.15}$$

We introduce finite-size correction terms $l^{(b,q)}(x)$ by writing $\mathfrak{a}^{(b,q)}(x)$ as

$$\mathfrak{a}^{(b,q)}(x) = \begin{cases} l^{(b,q)}(x) \; , & q \neq p \quad \text{or} \quad b \neq a \\ \tanh^N(nx/4) l^{(a,p)}(x) \; , & q = p \;\; \text{and} \;\; b = a \end{cases} \; . \tag{3.16}$$

For the ground state all the functions $l^{(b,q)}(x)$ are ANZC in $-\pi < x < \pi$. They satisfy the functional equations

$$\frac{l^{(b,q)}(x-\pi i/n)l^{(b,q)}(x+\pi i/n)}{l^{(b-1,q)}(x)l^{(b+1,q)}(x)} = \frac{\mathfrak{U}^{(b,q-1)}(x)\mathfrak{U}^{(b,q+1)}(x)}{\mathfrak{U}^{(b-1,q)}(x)\mathfrak{U}^{(b+1,q)}(x)} \; . \tag{3.17}$$

Again applying Fourier transforms to the logarithmic derivative of the equations the Fourier transform $L^{(b,q)}(k)$ to $l^{(b,q)}(x)$ satisfies

$$\begin{aligned} & L^{(b-1,q)}(k) - 2\cosh\Big(\frac{\pi k}{n}\Big) L^{(b,q)}(k) + L^{(b+1,q)}(k) \\ = \;& A^{(b-1,q)} + A^{(b+1,q)} - A^{(b,q+1)} - A^{(b,q-1)} \; . \end{aligned} \tag{3.18}$$

For fixed $q$ we have the closure conditions $L^{(0,q)} = L^{(n,q)} = A^{(0,q)} = A^{(n,q)} = 0$. This set of $n-1$ linear equations can be rewritten in matrix form as

$$\Big( Adj + K_0 \Big) \cdot L^q(k) = Adj \cdot A^q - A^{q+1} - A^{q-1} \tag{3.19}$$

with

$$K_0 = -2\cosh\Big(\frac{\pi k}{n}\Big) \; , \tag{3.20}$$

where $L^q(k)$ is $(n-1)\times 1$ matrix with the elements $L^{(b,q)}$ and the $(n-1)\times 1$ matrix $A^q(k)$ has the elements $A^{(b,q)}$. The matrix $Adj$ is the same as the adjacency matrix of the classical $A_{n-1}$ Dynkin diagram. The equations (3.19) are solvable. Similar equations in fact appear in [8] and these can be solved using a similar method. Thus, after integrating the equations, we obtain the set of nonlinear integral equations

$$l\mathfrak{a}^q = l\mathfrak{e}^q + K * l\mathfrak{U}^q - \hat{K} * \Big( l\mathfrak{U}^{q+1} + l\mathfrak{U}^{q-1} \Big) + D^q \,, \tag{3.21}$$

where the entries of the $(n-1)\times 1$ matrices $l\mathfrak{a}^q$, $lU^q$ and $l\mathfrak{e}^q$ are respectively given by $\ln \mathfrak{a}^{(b,q)}$, $\ln \mathfrak{U}^{(b,q)}$ and

$$\ln \mathfrak{e}^{(b,q)}(x) := \begin{cases} 0 \,, & q \neq p \quad \text{or} \quad b \neq a \\ \ln \tanh^N (nx/4) \,, & q = p \;\; \text{and} \;\; b = a \,. \end{cases} \tag{3.22}$$

The $(n-1)\times(n-1)$ matrices $K$ and $\hat{K}$ are the symmetric matrices whose entries in the upper right triangle are given by

$$\hat{K}_l^j(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\Big(\frac{\sinh[(n-l)\pi k/n]\sinh[j\pi k/n]}{\sinh[\pi k/n]\sinh(\pi k)}\Big) \tag{3.23}$$

$$K_l^j(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\Bigg( \coth\Big(\frac{\pi k}{n}\Big)\frac{\sinh[(n-l)\pi k/n]\sinh[j\pi k/n]}{\sinh(\pi k)}\Bigg). \tag{3.24}$$

The set of equations (3.21) are also obtained for the excited states after we take care of the extra zeros so that the ANZC properties hold in the analyticity strips.

## 3.2 Scaling limit

The finite-size corrections can be extracted from the nonlinear integral equations (3.21) and (3.12). The system size $N$ enters the nonlinear equations (3.21) through (3.22). The function $\mathfrak{e}^{(b,q)}$ has three asymptotic regimes with transitions in scaling regimes when $x$ is of the order of $-\ln N$ or $\ln N$. We suppose that $\mathfrak{a}^{(b,q)}$ and $\mathfrak{U}^{(b,q)}$ scale similarly and define:

$$\begin{aligned} e_{\pm}^{(b,q)}(x) &:= \lim_{N\to\infty} \mathfrak{e}^{(b,q)}\Big(\pm\frac{2}{n}(x+\ln N)\Big) \,, \\ a_{\pm}^{(b,q)}(x) &:= \lim_{N\to\infty} \mathfrak{a}^{(b,q)}\Big(\pm\frac{2}{n}(x+\ln N)\Big) \,, \end{aligned} \tag{3.25}$$

$$A_{\pm}^{(b,q)}(x) := \lim_{N\to\infty} \mathfrak{U}^{(b,q)}\Big(\pm\frac{2}{n}(x+\ln N)\Big) = 1 + a_{\pm}^{(b,q)}(x) \,, \tag{3.26}$$

In this scaling limit, (3.21) takes the form

$$\ell a^q = \ell e^q + K * \ell A^q - \hat{K} * \Big(\ell A^{q+1} + \ell A^{q-1}\Big) + D^q \,, \tag{3.27}$$

where we suppress the subscripts $\pm$. The entries of the $(n-1)\times 1$ matrices $\mathfrak{a}^q$, $U^q$ and $\mathfrak{e}^q$ are respectively given by $\ell a^{(b,q)}$, $\ell A^{(b,q)}$ and $\ell e^{(b,q)}$:

$$\ell a^{(b,q)}(x) \;:=\; \ln a^{(b,q)}(x) \,, \tag{3.28}$$

$$\ell A^{(b,q)}(x) \;:=\; \ln A^{(b,q)}(x) \,, \tag{3.29}$$

$$\ell e^{(b,q)}(x) \;:=\; \begin{cases} 0 \,, & q \neq p \quad \text{or} \quad b \neq a \,, \\ -2e^{-x} \,, & q = p \;\; \text{and} \;\; b = a \,, \end{cases} \tag{3.30}$$

where $q = 1, 2, \cdots, l-1$ and $b = 1, 2, \cdots, n-1$ and

$$a^{(b,q)}(x) = 0 \quad b < 1,\ b > n-1,\ q < 1 \text{ and } q > l-1 \tag{3.31}$$
$$\ell A^{(b,q)}(x) = 0 \quad b < 1,\ b > n-1,\ q < 1 \text{ and } q > l-1 \tag{3.32}$$

Now let us consider the scaling limit of $\mathfrak{b}^{(a,p)}(x)$, which gives the finite-size corrections to $\mathfrak{b}^{(a,p)}(x)$. Disregarding the integration constants in (3.12), and for fixed $x$, the finite-size corrections to $\mathfrak{b}^{(a,p)}(x)$ are given by the following expression of order $1/N$,

$$\begin{aligned}
&\ln \mathfrak{b}^{(a,p)}(x) := \lim_{N\to\infty} (k * \ln \mathfrak{U}^{(a,p)})(2x/n) \\
&= \frac{n}{4\pi} \lim_{N\to\infty} \int_{-\ln N}^{\infty} dy \Big( \frac{\ln \mathfrak{U}^{(a,p)}(y+\ln N)}{\cosh[x - n(y-\ln N)/2]} + \frac{\ln \mathfrak{U}^{(a,p)}(-y-\ln N)}{\cosh[x + n(y+\ln N)/2]} \Big) + \mathfrak{o}\Big(\frac{1}{N}\Big) \\
&= \frac{e^x}{N\pi} \int_{-\infty}^{\infty} e^{-y} \ell A_+^{(a,p)}(y)\, dy + \frac{e^{-x}}{N\pi} \int_{-\infty}^{\infty} e^{-y} \ell A_-^{(a,p)}(y)\, dy + \mathfrak{o}\Big(\frac{1}{N}\Big) \\
&= \frac{\cosh x}{N\pi} \int_{-\infty}^{\infty} e^{-y} \Big( \ell A_+^{(a,p)}(y) + \ell A_-^{(a,p)}(y) \Big)\, dy \\
&\quad + \frac{\sinh x}{N\pi} \int_{-\infty}^{\infty} e^{-y} \Big( \ell A_+^{(a,p)}(y) - \ell A_-^{(a,p)}(y) \Big)\, dy + \mathfrak{o}\Big(\frac{1}{N}\Big)\, .
\end{aligned} \tag{3.33}$$

It can be seen from (3.27) that for $q = p$ and $b = a$ the integrals in the above equation converge because the function $\ell A^{(a,p)}(y) = \ln\Big(1 + \ell a^{(a,p)}(y)\Big)$ tends to zero faster than any exponential. Moreover, as in the case of the critical ABF RSOS models, we have $\ell A_+^p(y) - \ell A_-^p(y) = 0$.

The integrals in (3.33) exist and can be evaluated explicitly with the help of the dilogarithmic function

$$L(x) = -\int_0^x dy\, \frac{\ln(1-y)}{y} + \tfrac{1}{2} \ln x \ln(1-x). \tag{3.34}$$

To show this, multiplying the derivative of (3.16) with $\ell A^{(b,q)}$, and (3.16) itself with $(\ell A^{(b,q)})'$, then taking the difference and summing over $q$ and $b$, and finally integrating we derive

$$\begin{aligned}
&\sum_{q=1}^{l-1} \sum_{b=1}^{n-1} \int_{-\infty}^{\infty} [(\ell a^{(b,q)})' \ell A^{(b,q)} - \ell a^{(b,q)} (\ell A^{(b,q)})']\, dx = \\
&\sum_{q=1}^{l-1} \sum_{b=1}^{n-1} \int_{-\infty}^{\infty} [(\ell e^{(b,q)})' \ell A^{(b,q)} - (\ell e^{(b,q)} - D^{(b,q)}) (\ell A^{(b,q)})']\, dx\, ,
\end{aligned} \tag{3.35}$$

where the contribution of the kernel cancels due to the symmetry

$$k(-x) = k(x)\, . \tag{3.36}$$

Then using the nonlinear integral equations (3.21) to simplify the right-hand side and integrating the left-hand side of (3.35), we can write the finite-size corrections as

$$\begin{aligned}
&2 \int_{-\infty}^{\infty} e^{-y} \ell A^{(a,p)}(y)\, dy = \\
&- \sum_{q=1}^{l-1} \sum_{b=1}^{n-1} L\Big(\frac{1}{A^{(b,q)}}\Big)\Big|_{-\infty}^{\infty} + \tfrac{1}{2} \sum_{q=1}^{l-1} \sum_{b=1}^{n-1} D^{(b,q)} \ell A^{(b,q)} \Big|_{-\infty}^{\infty}
\end{aligned} \tag{3.37}$$

where the constants $D^{(b,q)}=0$ which can be shown by the asymptotics of the equations (3.27).

It can be seen from (3.37) that the finite-size corrections of the transfer matrices in the scaling limit depend only on the solutions in the limits $x\to\pm\infty$, which are given by the asymptotics and bulk behavior.

## 3.3 The central charge and the conformal weights

The following useful dilogarithm identity has been established by Kirillov [23]. Consider the functions

$$y^{(b,q)}(j,n,r):=\frac{\sin[(q+n)\varphi]\sin[q\varphi]}{\sin[a\varphi]\sin[(n-a)\varphi]}\,,\qquad 1\le b\le n-1,\qquad 1\le q\le r \tag{3.38}$$

with

$$\varphi=\frac{(1+j)\pi}{n+r}\qquad 0\le j\le n+r-2\,. \tag{3.39}$$

It is obvious that they are the asymptotic solutions of the functions equations (2.14) with $r=l$ or the bulk behavior of the functions equations with $r=p$ and $r=l-p$. Then the following dilogarithmic function identity holds,

$$\begin{aligned} s(j,n,l) \;&:=\; \sum_{k=1}^{n-1}\sum_{m=1}^{r} L(\frac{1}{1+y^{(k,m)}(j,n,r)}) \\ &=\; L(1)\Big(\frac{(n^2-1)r}{n+r}-\frac{n(n^2-1)j(j+2)}{n+r} \\ &\quad +6j\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n+1}{2}\right\rfloor+6\mathbb{Z}_+\Big)\,. \end{aligned} \tag{3.40}$$

Here the brackets $\lfloor x\rfloor$ denotes the greatest integer less than or equal to $x$.

The finite-size corrections (3.33) are expressed in terms of the dilogarithm function in (3.37)

$$\ln\mathfrak{b}^{(a,p)}(x)=-\frac{\cosh x}{N\pi}\sum_{q=1}^{l-1}\sum_{b=1}^{n-1}L\Big(\frac{1}{A^{(b,q)}}\Big)\Big|_{-\infty}^{\infty}\,. \tag{3.41}$$

The inputs are the asymptotic and bulk solutions obtained in sections 2.4 and 2.6. Thus we have

$$\begin{aligned} \ln\mathfrak{b}^{(a,p)}(x) \;&=\; \frac{\cosh x}{N\pi}\left(\sum_{b=1}^{n-1}\sum_{q=1}^{l-1}L\Big(\frac{1}{A^{(b,q)}(-\infty)}\Big)-\sum_{b=1}^{n-1}\sum_{q=1}^{l-1}L\Big(\frac{1}{A^{(b,q)}(\infty)}\Big)\right) \\ &=\; \frac{\cosh x}{N\pi}\left(\sum_{b=1}^{n-1}\sum_{q=1}^{l}L\Big(\frac{1}{A^{(b,q)}(-\infty)}\Big)-\sum_{b=1}^{n-1}\sum_{q=1}^{l}L\Big(\frac{1}{A^{(b,q)}(\infty)}\Big)\right) \\ &=\; \frac{\cosh x}{N\pi}\left(\sum_{b=1}^{n-1}\left[\sum_{q=1}^{l-p-1}L\Big(\frac{1}{A^{(b,q)}(-\infty)}\Big)+L\Big(\frac{1}{A^{(b,p)}(-\infty)}\Big)\right]\right. \\ &\quad \left.+\sum_{b=1}^{n-1}\sum_{q=p+1}^{l}L\Big(\frac{1}{A^{(b,q)}(-\infty)}\Big)-\sum_{b=1}^{n-1}\sum_{q=1}^{l}L\Big(\frac{1}{A^{(b,q)}(\infty)}\Big)\right)\,. \end{aligned} \tag{3.42}$$

With the help of the dilogarithm identity the finite-size corrections can be expressed as

$$\begin{aligned}
\ln \mathfrak{b}^{(a,p)}(x) &= \frac{\cosh x}{N\pi}\left(s(m',n,p)+s(m'',n,h-n-p)-s(m,n,h-n)\ \right)\\
&= \frac{\pi\cosh x}{6N}\Big(\frac{(n^2-1)(h-n-p)}{h-p}-\frac{n(n^2-1)m''(m''+2)}{h-p}\\
&\quad+\frac{(n^2-1)p}{n+p}-\frac{n(n^2-1)m'(m'+2)}{n+p}\\
&\quad-\frac{(n^2-1)(h-p)}{h}+\frac{n(n^2-1)m(m+2)}{h}\\
&\quad+6(m'+m''-m)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n+1}{2}\right\rfloor+6\mathbb{Z}\Big)\ . \qquad (3.43)
\end{aligned}$$

For the excited states $m'', m', m$ are greater than 1 and also no longer independent. We suppose that

$$m' = m - m'' + n\left\lfloor\frac{m-m''}{p}\right\rfloor \qquad (3.44)$$

The relation (3.44) among $m'', m', m$ has been shown for $n = 2$ in [8] and for $n \geq 2$ in [16]. Using this relation and comparing (3.43) with the finite-size correction

$$\ln \mathfrak{b}^{p}(x) = \frac{\pi}{6N}\Big(c-12(\Delta+\overline{\Delta})+(k+\overline{k})\Big)\cosh x+\mathfrak{o}\Big(\frac{1}{N}\Big), \qquad (3.45)$$

(3.43) yields the central charge

$$c = \frac{(n^2-1)p}{n+p}-\frac{n(n^2-1)p}{h(h-p)} \qquad (3.46)$$

and the conformal weights

$$\Delta_{t,s} = \frac{n(n^2-1)\{[ht-(h-p)s]^2-p^2\}}{24ph(h-p)}-\frac{n(n^2-1)\nu(n\nu-2p)}{24p(p+n)}\ , \qquad (3.47)$$

where the exponents $s = m$ and $t = m''$ are integers satisfying

$$1 \leq s \leq \left\lfloor\frac{l}{n-1}\right\rfloor\ ,\qquad 1\leq t\leq\left\lfloor\frac{l-p}{n-1}\right\rfloor\ ,\qquad p=1,2,\cdots,h-2 \qquad (3.48)$$

and

$$\nu := (s-t)-\left\lfloor\frac{s-t}{p}\right\rfloor p\ . \qquad (3.49)$$

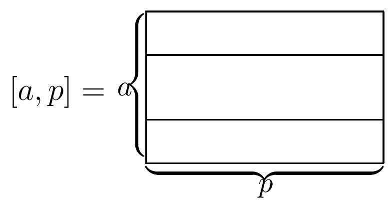
The integers $k$ and $\overline{k}$ are such that

$$\begin{aligned}
k+\overline{k} &= n(n^2-1)\left\lfloor\frac{s-t}{p}\right\rfloor\Big(n\left\lfloor\frac{s-t}{p}\right\rfloor+2\Big)+6n\left\lfloor\frac{s-t}{p}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n+1}{2}\right\rfloor+6\mathbb{Z}\\
&= 0 \quad \mathrm{mod}\ 6 \qquad (3.50)
\end{aligned}$$

do not contribute to the central charge and conformal weights and hence can be discarded.

# 4 Discussion

We have presented the results for the finite-size corrections of transfer matrices for fused $A_{n-1}^{(1)}$ models of Jimbo, Miwa and Okado. These calculations generalize the evaluation of the finite-size corrections of transfer matrices for the fused ABF models given in [8] and the central charge calculation of the $A_{n-1}^{(1)}$ models given in [19]. In this paper we have studied the fused transfer matrices with rectangular Young diagrams




These transfer matrices satisfy the functional equations (2.12). The finite-size corrections of the transfer matrices with the fusion of Young diagram $[a,p]$ in vertical direction are expressed by (3.43). Using Young diagrams the result (3.43) can be represented by

$$\sum_{b=1}^{n-1}\Big([b,l+n-p]+[b,p+n]-[b,l+n]\Big) \tag{4.51}$$

Suppose that the solution (3.38) corresponds to the Young diagram $[b,q]$. The expression (4.51) gives the Rogers dilogarithm explanation for the central charges (1.11) and the conformal weights (1.14), which are missed in [23]. The expression (4.51) also shows the coset structure corresponding to the central charge (1.11) and the conformal weights (1.12) with the case of (1.15). For other cases the similar coset has been conjectured [16] and need to be confirmed, which have not been done in this paper.

The conformal weights obtained in this paper are a subset of all possible conformal weights (1.12). But our calculation has shown that in order to have all conformal weights we have to analyze the complex eigenvalues of the inversion identity hierarchies of the models. The difficulty part is to find the related dilogarithm identity involving the complex eigenvalues. These interesting questions are left for further investigation.

The method by solving the functional equations of fusion hierarchies to find the finite size corrections of the eigenvalues of the fused transfer matrices firstly was developed in [8] for study of the fused ABF restricted SOS models and more recently for the central charges of the fused $A_{n-1}^{(1)}$ models [19]. In fact, the functional equations of the fused transfer matrices contain enough information to calculate the eigenvalues of the transfer matrices. Hence the eigenvalues and the relevant Bethe ansatz equations can be extracted from the functional equations and the finite size corrections can be calculated by solving the Bethe ansatz equations [7, 17]. These two methods technically are different, solving Bethe ansatz one employs the string hypothesis and the other one relies on the ANZC property. But it seems that by solving the functional equations it is more powerful to find the conformal weights. Our present paper has shown this partially for the fused $A_{n-1}^{(1)}$ models. There are a large group of fused transfer matrices for the higher rank $n>2$. For example, the fused transfer matrices with other Young diagrams. These have not been

evaluated here. But the finite-size corrections of these transfer matrices should fit the picture (4.51). However, we need other functional equations to solve the fused transfer matrices with other Young diagrams.

There are many solvable IRF models, e.g. $B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ models in [40], $A_n^{(2)}$ models in [41] and dilute $A_L$ models in [42]. Also we have the functional equations for the fused transfer matrices of the models [18, 19, 43, 44, 45]. It should be possible to solve these functional equations by similar methods.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## Appendix A: The adjacency matrices of the fused JMO $A_{n-1}^{(1)}$ models for $n = 4$ and $l = 2$.

The set of dominant integrable weights $P_+(4,2)$ is given by [(0,0,0),(1,0,0),(0,1,0),(0,0,1),(2,0,0),(1,1,0),(1,0,1),(0,2,0),(0,1,1),(0,0,2)] according to the Young diagram notation (see Figure 2). The adjacency matrices of the model with rank $n = 4$ and level $l = 2$ is labeled by the order of the elements in $P_+(4,2)$ for the rows and the columns of matrix.

$$
\begin{aligned}
A^{(0,0,0)} &= 1 \\
A^{(m,n,l)} &= 0 \quad \text{for } m+n+l > 2 \text{ and } m,n,l < 0
\end{aligned}
$$

$$
A^{(0,0,1)} = \begin{pmatrix}
0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&1&0
\end{pmatrix}
\qquad
A^{(0,1,0)} = \begin{pmatrix}
0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0\\
1&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&1&0&1&0&0&0&0&1\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0
\end{pmatrix}
$$

$$
A^{(0,1,1)} = \begin{pmatrix}
0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&1\\
0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0\\
0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0
\end{pmatrix}
\qquad
A^{(1,0,0)} = \begin{pmatrix}
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0\\
1&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&0
\end{pmatrix}
$$

$$
A^{(1,1,0)} = \begin{pmatrix}
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&1\\
0&0&0&1&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&0&0
\end{pmatrix}
\qquad
A^{(1,0,1)} = \begin{pmatrix}
0&0&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&1&0&1&0&0&0&0&1\\
0&0&0&1&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0\\
1&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$A^{(2,0,0)} = \begin{pmatrix} 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0 \end{pmatrix} \qquad A^{(0,2,0)} = \begin{pmatrix} 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0 \end{pmatrix}$$

$$A^{(0,0,2)} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0 \end{pmatrix}$$

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